

INTEGRAL AUTOMORPHISMS OF AFFINE SPACES OVER FINITE FIELDS

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ABSTRACT. A permutation of the point set of the affine space $\text{AG}(n, q)$ is called an integral automorphism if it preserves the integral distance defined among the points. In this paper, we complete the classification of the integral automorphisms of $\text{AG}(n, q)$ for $n \geq 3$.

1. INTRODUCTION

Throughout the paper p stands for an odd prime. Let \mathbb{F}_q be the finite field with $q = p^h$ elements and $\text{AG}(n, q)$ be the n -dimensional affine space defined over \mathbb{F}_q . The *Euclidean distance* d is defined as

$$d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n (x_i - y_i)^2$$

for the points $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n) \in \mathbb{F}_q^n$. Two points \mathbf{x} and \mathbf{y} are said to be at *integral distance* if $d(\mathbf{x}, \mathbf{y})$ is a square element in \mathbb{F}_q including the zero element, and a set of points is called *integral* if any two of its points are at integral distance. Recently, the finite field analog of the classical problem about integral point sets in \mathbb{R}^n has attracted considerable attention. See, for example, [4] and the references therein. Besides integral point sets, permutations, preserving integral distances, are also considered in [6, 7, 8, 9]. By an *integral automorphism* of $\text{AG}(n, q)$ we mean any bijective mapping $\gamma : \mathbb{F}_q^n \rightarrow \mathbb{F}_q^n$ satisfying

$$d(\mathbf{x}, \mathbf{y}) \in \square_q \iff d(\mathbf{x}^\gamma, \mathbf{y}^\gamma) \in \square_q$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n$. Here and in what follows \square_q denotes the set of all square elements of \mathbb{F}_q including the zero element. We adopt the notation used in [6] and denote the group of all integral automorphisms by $\text{Aut}(\mathbb{F}_q^n)$.

Integral automorphisms of the plane $\text{AG}(2, q)$ were determined in [6, 7, 8]. In particular, $\text{Aut}(\mathbb{F}_q^2)$ was found by Kurz [8] for $q \equiv 3 \pmod{4}$, and by Kovács and Ruff [7] for $q \equiv 1 \pmod{4}$. We remark that the special case $q = p$ was settled earlier by Kiermaier and Kurz [6]. It turns out that there exist integral automorphisms of $\text{AG}(2, q)$ which are not semiaffine transformations, and this occurs exactly when $q \equiv 1 \pmod{4}$. As for higher dimensions, Kurz and Meyer [9] described the integral automorphisms which are also semiaffine transformations. In what follows we denote by \mathbb{F}_q^\times the multiplicative group of \mathbb{F}_q , by $\text{GL}(n, q)$ the group of invertible $n \times n$ matrices with entries from \mathbb{F}_q , and by σ the semiaffine transformation defined by $(x_1, \dots, x_n) \mapsto (x_1^p, \dots, x_n^p)$.

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Theorem 1 (Kurz and Meyer [9]). *If $q = p^h$ and $n \geq 3$, then the semiaffine transformations contained in $\text{Aut}(\mathbb{F}_q^n)$ are given as*

$$\mathbf{x} \mapsto a\mathbf{x}^{\sigma^i}A + \mathbf{b}$$

where $a \in \mathbb{F}_q^\times$, $i \in \{0, \dots, h-1\}$, $A \in \text{GL}(n, q)$ with $AA^T = I$ and $\mathbf{b} \in \mathbb{F}_q^n$.

Our goal in this paper is to show that, in contrast with the plane, all integral automorphisms of $\text{AG}(n, q)$ are semiaffine transformations whenever $n \geq 3$. This together with Theorem 1 result in the following classification theorem.

Theorem 2. *Let $q = p^h$ for an odd prime p and suppose that $n \geq 3$. Then the integral automorphisms of $\text{AG}(n, q)$ are the mappings*

$$\mathbf{x} \mapsto a\mathbf{x}^{\sigma^i}A + \mathbf{b}$$

where $a \in \mathbb{F}_q^\times$, $i \in \{0, \dots, h-1\}$, $A \in \text{GL}(n, q)$ with $AA^T = I$ and $\mathbf{b} \in \mathbb{F}_q^n$.

2. PROOF OF THEOREM 2

The key part in the proof of Theorem 2 will be to show that every integral automorphism $\gamma \in \text{Aut}(\mathbb{F}_q^n)$ satisfies

$$d(\mathbf{x}, \mathbf{y}) = 0 \iff d(\mathbf{x}^\gamma, \mathbf{y}^\gamma) = 0 \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{F}_q^n. \quad (1)$$

This enables us to use the result of Lester [11] about cone preserving mappings. Let V be a nonsingular metric vector space over a field \mathbb{F} whose characteristic is different from two, upon which is defined a nonsingular symmetric bilinear form $\langle \cdot, \cdot \rangle$. The cone $C(\mathbf{a})$ with vertex $\mathbf{a} \in V$ is defined to be the set $C(\mathbf{a}) := \{\mathbf{x} \in V : \langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = 0\}$, and a mapping $f : V \rightarrow V$ is said to *preserve cones* if $(C(\mathbf{a}))^f = C(\mathbf{a}^f)$ for all $\mathbf{a} \in V$.

Theorem 3 (Lester [11]). *Let V be a nonsingular metric vector space over the field \mathbb{F} , with bilinear form $\langle \cdot, \cdot \rangle$; assume that $\dim(V) \geq 3$ and that V is not anisotropic (that is, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ for some nonzero vector \mathbf{x}). Let $f : V \rightarrow V$ be a bijection of V which preserves cones. Then f is of the form*

$$f : \mathbf{x} \mapsto L(\mathbf{x}) + \mathbf{b}$$

where $\mathbf{b} \in V$, and (L, ρ) is a semilinear transformation of V satisfying $\langle L(\mathbf{x}), L(\mathbf{y}) \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle^\rho$ for some nonzero $a \in \mathbb{F}$ and for all $\mathbf{x}, \mathbf{y} \in V$.

Now, if $\gamma \in \text{Aut}(\Gamma)$ satisfies (1), then it preserves cones of the metric vector space $V := \mathbb{F}_q^n$ equipped with the symmetric bilinear form $\langle \cdot, \cdot \rangle$ defined by $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{x}\mathbf{y}^T$ for all vectors $\mathbf{x}, \mathbf{y} \in V$. Therefore, by Theorem 3, γ is a semiaffine transformation, and Theorem 2 follows. In fact, we are going to derive (1) in the end of this section following two preparatory lemmas.

For the rest of the paper we let $G = \text{Aut}(\mathbb{F}_q^n)$, $n \geq 3$, and let G_0 be the stabilizer of $\mathbf{0}$ in G where $\mathbf{0} = (0, \dots, 0) \in \mathbb{F}_q^n$. We start by introducing two subgroups of G :

$$\begin{aligned} E &= \{\mathbf{x} \mapsto \mathbf{x} + \mathbf{b} : \mathbf{b} \in \mathbb{F}_q^n\}, \\ M &= \{\mathbf{x} \mapsto a\mathbf{x}A : a \in \mathbb{F}_q^\times, A \in \text{GL}(n, q) \text{ and } AA^T = I\}. \end{aligned}$$

Notice that, by Theorem 1, both E and M are subgroups of G . The elements of E are also called *translations*. Clearly, E is an elementary abelian group of order p^{hn} , and it is regular on \mathbb{F}_q^n . The group M normalizes E , hence $\langle E, M \rangle = EM$.

Define the subsets of \mathbb{F}_q^n as

$$\begin{aligned} S_0 &= \left\{ \mathbf{x} \in \text{AG}(n, q) : \sum_{i=1}^n x_i^2 = 0, \mathbf{x} \neq \mathbf{0} \right\}, \\ S_+ &= \left\{ \mathbf{x} \in \text{AG}(n, q) : \sum_{i=1}^n x_i^2 \in \square_q \setminus \{0\} \right\}, \\ S_- &= \left\{ \mathbf{x} \in \text{AG}(n, q) : \sum_{i=1}^n x_i^2 \notin \square_q \right\}. \end{aligned}$$

Lemma 1. *With the above notation,*

- (i) *The M -orbits are $\{\mathbf{0}\}, S_0, S_+$ and S_- .*
- (ii) *EM is primitive on \mathbb{F}_q^n .*

Proof. Part (i) is proved in [9, Lemma 3.17].

To settle (ii) we apply [1, Theorem 3.2A], that is, EM is primitive if and only if $\text{Graph}(\Delta)$ is connected for each nondiagonal orbital Δ of EM . Observe that, a nondiagonal orbital Δ consists of the ordered pairs in the form $(\mathbf{x}, \mathbf{x} + \mathbf{y})$, where \mathbf{x} runs over \mathbb{F}_q^n and \mathbf{y} runs over S_ε for a fixed $\varepsilon \in \{0, +, -\}$. Now, the connectedness of $\text{Graph}(\Delta)$ follows because each of S_0, S_+ and S_- spans the vector space \mathbb{F}_q^n . \square

By Lemma 1(i), EM has nontrivial subdegrees $|S_\varepsilon|, \varepsilon \in \{0, +, -\}$. The exact values were computed in [9, Theorem 4.3]:

$$|S_0| = \begin{cases} q^{n-1} - 1 & \text{if } n \text{ is odd} \\ q^{n-1} + (-1)^{\frac{\varepsilon n}{2}} q^{\frac{n}{2}} - (-1)^{\frac{\varepsilon n}{2}} q^{\frac{n-2}{2}} - 1 & \text{if } n \text{ is even} \end{cases} \quad (2)$$

$$|S_+| = \begin{cases} \frac{1}{2} \left(q^n - q^{n-1} + (-1)^{\frac{\varepsilon(n+3)}{2}} q^{\frac{n+1}{2}} - (-1)^{\frac{\varepsilon(n-1)}{2}} q^{\frac{n-1}{2}} \right) & \text{if } n \text{ is odd} \\ \frac{1}{2} \left(q^n - q^{n-1} - (-1)^{\frac{\varepsilon n}{2}} q^{\frac{n}{2}} + (-1)^{\frac{\varepsilon n}{2}} q^{\frac{n-2}{2}} \right) & \text{if } n \text{ is even} \end{cases} \quad (3)$$

$$|S_-| = \begin{cases} \frac{1}{2} \left(q^n - q^{n-1} - (-1)^{\frac{\varepsilon(n+3)}{2}} q^{\frac{n+1}{2}} + (-1)^{\frac{\varepsilon(n-1)}{2}} q^{\frac{n-1}{2}} \right) & \text{if } n \text{ is odd} \\ \frac{1}{2} \left(q^n - q^{n-1} - (-1)^{\frac{\varepsilon n}{2}} q^{\frac{n}{2}} + (-1)^{\frac{\varepsilon n}{2}} q^{\frac{n-2}{2}} \right) & \text{if } n \text{ is even} \end{cases} \quad (4)$$

where $\varepsilon = 0$ if $q \equiv 1 \pmod{4}$ and $\varepsilon = 1$ otherwise.

The set $S_0 \cup S_+$ consists of the points being at integral distance from $\mathbf{0}$. Therefore, every $\gamma \in G_{\mathbf{0}}$ maps $S_0 \cup S_+$ to itself, and this together with Lemma 1(i) leave us with two possibilities for the nontrivial $G_{\mathbf{0}}$ -orbits, namely, these are either S_0, S_+ and S_- , or $S_0 \cup S_+$ and S_- . In particular, the group G has rank either 3 with nontrivial subdegrees $|S_0| + |S_+|$ and $|S_-|$, or 4 with nontrivial subdegrees $|S_0|, |S_+|$ and $|S_-|$.

As the next step, we find the *socle* $\text{Soc}(G)$. Recall that $\text{Soc}(G)$ is the subgroup of G generated by all its minimal normal subgroups.

Lemma 2. *With the above notation, the socle $\text{Soc}(G) = E$.*

Proof. Let $H = \text{Soc}(G)$. Since $EM \leq G$ is primitive, see Lemma 1(ii), G is primitive as well. Thus H is a direct product of isomorphic simple groups (see [1, Corollary 4.3B]), and we may write $H = T \times \cdots \times T = T^k$ for some simple group T and $k \geq 1$. By the O’Nan-Scott theorem, G and H are described by one of the following types (see, for example, [1, pp. 137]):

- (T1) H is an elementary abelian p -group of order q^n which is regular on \mathbb{F}_q^n .
- (T2) H is nonabelian and regular on \mathbb{F}_q^n .
- (T3) $H = T$ is nonabelian, it is not regular on \mathbb{F}_q^n , and $G \leq \text{Aut}(H)$.
- (T4) H is nonabelian and G is a subgroup of a wreath product with the diagonal action. In this case $k \geq 2$ and $|T|^{k-1} = q^n$.
- (T5) H is nonabelian, $k = k_1 k_2$ and $k_2 > 1$. The group G is isomorphic to a subgroup of the wreath product $U \text{ wr } S_{k_2}$ with the product action, where U is a primitive permutation group of degree d such that $q^n = d^{k_2}$, U has socle T^{k_1} , and U is of type (T3) or (T4).

We show below that G is of type (T1). It is not hard to show that this yields $H = E$ (see, for example, [7]). Now, suppose to the contrary that G is described by one of types (T2) – (T5). In each case T is a nonabelian simple group. This observation excludes at once types (T2) and (T4).

Suppose next that G is of type (T3). Then $T = H$, and since it is a normal subgroup of a primitive group, it acts transitively on \mathbb{F}_q^n . It was proved by Guralnick [3] that, if a finite nonabelian simple group L acts transitively on a set Ω such that $|\Omega|$ is a prime power, then L acts 2-transitively unless $L \cong \text{PSU}(4, 2)$ and $|\Omega| = 27$ with nontrivial subdegrees 10 and 16 (see [3, Corollary 2]). Since G cannot be 2-transitive, $q^n = 27$ and the nontrivial subdegrees of G are 10 and 16. This, however, contradicts that $|S_-| = 12$ is a subdegree, see the remark before the lemma and (4).

We are left with the case that G is of type (T5). Denote by r_G and r_U the rank of G and U , respectively. Recall that $r_G \in \{3, 4\}$. By [1, Exercise 4.8.1],

$$r_G \geq \binom{r_U + k_2 - 1}{k_2}. \quad (5)$$

The group U is of type (T3) or (T4). In the latter case $|T| = p^a$ for some a , a contradiction. Thus U is of type (T3), $k_1 = 1$, $k = k_2$ and T is a transitive permutation group of a set X of size $|X| = q^{n/k_2}$. By the aforementioned result of Guralnick, U is 2-transitive unless $T \cong \text{PSU}(4, 2)$, $q^{n/k_2} = 27$, and $r_U = 3$. In the latter case, however, we find in (5) that $r_G \geq \frac{1}{2}(k_2 + 2)(k_2 + 1) \geq 6$ (recall that $k_2 > 1$), a contradiction. Thus $r_U = 2$, implying in (5) that $k = k_2 = 2$ and $r_G = 3$, or $k = k_2 = 3$ and $r_G = 4$.

Case 1. $k_2 = 2$, $r_G = 3$ and $G \leq U \text{ wr } S_2$.

The wreath product $U \text{ wr } S_2$ acts by the product action. This means that \mathbb{F}_q^n can be written as $\mathbb{F}_q^n = X \times X$, $|X| = q^{n/2}$, and U is a permutation group of X . We have $U \text{ wr } S_2 = \langle U \times U, \tau \rangle = \langle U \times U \rangle \rtimes \langle \tau \rangle$, where $U \times U$ acts on $X \times X$ naturally, and τ acts by switching the coordinates. The socle $H = T \times T \leq U \times U$, and since T is 2-transitive on X , $\Delta_1 := \{(x_0, x) : x \in X \setminus \{x_0\}\}$ and $\Delta_2 := \{(x, x_0) : x \in X \setminus \{x_0\}\}$ are orbits under the stabilizer $(U \times U)_{(x_0, x_0)}$, and any other orbit different from $\{(x_0, x_0)\}$ is contained in the set $\Delta_3 := \{(x, y) : x, y \in X \setminus \{x_0\}\}$. Now, $G_{(x_0, x_0)} = (U \times U)_{(x_0, x_0)} \rtimes \langle \tau \rangle$,

and this gives that any $G_{(x_0, x_0)}$ -orbit different from $\{(x_0, x_0)\}$ is contained in either $\Delta_1 \cup \Delta_2$ or Δ_3 . Since the rank $r_G = 3$, we find that the nontrivial subdegrees of G are $|\Delta_1 \cup \Delta_2| = 2(q^{n/2} - 1)$ and $|\Delta_3| = (q^{n/2} - 1)^2$. On the other hand $|S_-|$ is a subdegree which is divisible by q , see (4) (we use here that $n \geq 3$).

Case 2. $k_2 = 3$, $r_G = 4$ and $G \leq U \wr S_3$.

In this case \mathbb{F}_q^n can be written as $\mathbb{F}_q^n = X \times X \times X$, $|X| = q^{n/3}$, and U is a permutation group of X . The wreath product $U \wr S_3 = \langle U \times U \times U \rangle \rtimes K$, where $U \times U \times U$ acts on $X \times X \times X$ naturally, $K \cong S_3$, and K acts by permuting the coordinates. The socle $H = T \times T \times T \leq U \times U \times U$ and T is 2-transitive on X . Now, $G_{(x_0, x_0, x_0)} \leq (U \times U \times U)_{(x_0, x_0, x_0)} \rtimes K$, and this gives that any $G_{(x_0, x_0, x_0)}$ -orbit different from $\{(x_0, x_0, x_0)\}$ is contained in one of the sets $\{(x, x_0, x_0), (x_0, x, x_0), (x_0, x_0, x) : x \in X \setminus \{(x_0, x_0, x_0)\}\}$, $\{(x, y, x_0), (x, x_0, y), (x_0, x, y) : x, y \in X \setminus \{(x_0, x_0, x_0)\}\}$ and $\{(x, y, z) : x, y, z \in X \setminus \{(x_0, x_0, x_0)\}\}$. Because of this and $r_G = 4$ we find that the nontrivial subdegrees of G are $3(q^{n/3} - 1)$, $3(q^{n/3} - 1)^2$ and $(q^{n/3} - 1)^3$. On the other hand these subdegrees are $|S_\varepsilon|$, $\varepsilon \in \{0, +, -\}$, and as $q^{\lceil \frac{n-2}{2} \rceil}$ divides both $|S_+|$ and $|S_-|$ and hn is divisible by 3, we obtain that $(q, n) = (3, 3)$, and therefore, $U \cong S_3$ and $T \cong \mathbb{Z}_3$, contradicting that T is nonabelian. \square

We are ready to settle (1).

Lemma 3. *Let $\gamma \in \text{Aut}(\mathbb{F}_q^n)$ be an arbitrary automorphism and let $n \geq 3$. Then γ satisfies (1).*

Proof. Suppose for the moment that $q = p$. By Lemma 2, $E = \text{Soc}(G)$, in particular, E is normal in G . Now, since $q = p$, we obtain that γ is an affine transformation, and this implies that it satisfies (1).

From now on it will be assumed that $q \neq p$. Assume to the contrary that there exist vectors \mathbf{a} and \mathbf{b} such that either $d(\mathbf{a}, \mathbf{b}) = 0$ and $d(\mathbf{a}^\gamma, \mathbf{b}^\gamma) \neq 0$, or $d(\mathbf{a}, \mathbf{b}) \neq 0$ and $d(\mathbf{a}^\gamma, \mathbf{b}^\gamma) = 0$. Here we deal only with the first case because the second one can be treated in a very similar way. Consider the product $\gamma' := \gamma_1 \gamma \gamma_2$ where γ_1 and γ_2 are the translations $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$ and $\mathbf{x} \mapsto \mathbf{x} - \mathbf{a}^\gamma$, respectively. Then $\mathbf{0}^{\gamma'} = \mathbf{0}$, $\mathbf{b} - \mathbf{a} \in S_0$, and $(\mathbf{b} - \mathbf{a})^{\gamma'} = \mathbf{b}^\gamma - \mathbf{a}^\gamma \in S_+$. These imply that the G_0 -orbits are $\{\mathbf{0}\}$, $S_0 \cup S_+$ and S_- (see also the remark before Lemma 2), and thus G has nontrivial subdegrees:

$$|S_0| + |S_+| \text{ and } |S_-|. \quad (6)$$

By Lemma 2, G is of type (T1), that is, it is an affine permutation group. Therefore, we can use the classification of finite primitive affine permutation groups of rank 3 due to Liebeck (see [10, Theorem 1.1]). Namely, if L is such a group acting on a vector space V of cardinality p^d , and L_0 denotes the stabilizer of the zero vector 0 , then one of the following holds:

Infinite classes (A): L is in one of 11 infinite classes of permutation groups labeled by (A1)–(A11). If L is in class (A1), then L_0 is isomorphic to a subgroup of $\Gamma\text{L}(1, p^d)$; and if L is in class (A2)–(A11), then $d = 2r$ and L has nontrivial subdegrees listed in Table 2 (see [10, Table 12]).

row	subdegrees	conditions
1.	$(p^s + 1)(p^r - 1), \quad p^s(p^r - 1)(p^{r-s} - 1)$	$s = 0$ or $s \mid r$ or $s = 2r/5$ and $5 \mid r$ or $s = 3r/4$ and $4 \mid r$ or $s = 3r/8$ and $8 \mid r$
2.	$(p^{r-s} + 1)(p^r - 1), \quad p^{r-s}(p^r - 1)(p^s - 1)$	$s \mid r$
3.	$(p^{r-s} - 1)(p^r + 1), \quad p^{r-s}(p^r + 1)(p^s - 1)$	$s \mid r$ and $s \neq r$

TABLE 1. Nontrivial subdegrees of affine groups of rank 3 in classes (A2)–(A11).

‘*Extraspecial*’ classes (B): L is one of a finite set of permutation groups whose degree is equal to one of the following numbers ([10, Table 1]):

$$2^6, 3^4, 3^6, 3^8, 5^4, 7^2, 7^4, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2, 47^2. \quad (7)$$

‘*Exceptional*’ classes (C): L is one of a finite set of permutation groups whose degree is equal to one of the following numbers ([10, Table 2]):

$$2^6, 2^8, 2^{11}, 2^{12}, 3^4, 3^5, 3^6, 3^{12}, 5^4, 5^6, 7^4, 31^2, 41^2, 71^2, 79^2, 89^2. \quad (8)$$

We are going to arrive at a contradiction after comparing the subdegrees described in classes (A)–(C) with our subdegrees in (6).

Suppose that G is in class (A). If G is in class (A1), then G_0 is isomorphic to a subgroup of $\Gamma L(1, q^n)$, hence $|G_0|$ divides $|\Gamma L(1, q^n)| = hn(q^n - 1)$. Each subdegree of G divides $|G_0|$. In particular, $|S_-| \mid |G_0|$, and by (4), $p^{h\lceil \frac{n-2}{2} \rceil} \mid hn(q^n - 1)$. From this we obtain that $p^m \leq 4m$ where p is an odd prime and $m = h\lceil \frac{n-2}{2} \rceil \geq 2$ (recall that $n \geq 3$ and $h \geq 2$ because of $q \neq p$). This, however, contradicts the inequality $p^m > 4m$, which can be easily settled by induction on m .

Let G be in class (Ai) for $i > 1$. As before, let $m = h\lceil \frac{n-2}{2} \rceil$. By (4), p^m is the largest p -power dividing the subdegree $|S_-|$, and we get $2|S_-|/p^m \equiv \pm 1 \pmod{q}$. Thus

$$2|S_-|/p^m \equiv \pm 1 \pmod{p^2}. \quad (9)$$

Let us compute the residue of $2|S_-|/p^m$ modulo p^2 by the help of Table 1. Since $q^n = p^{2r}$, it follows that $2r = hn$, and hence $r \geq 3$. Suppose that $|S_-|$ occurs in the 1st row of Table 1. In this case $m = s$, and hence $s \geq 2$. Notice that, if $s \mid r$, then $s \mid (r - s)$, from which $r - s \geq 2$ and this implies that $2|S_-|/p^m \equiv 2 \pmod{p^2}$, contradicting (9). Now, it follows that $r - s \geq 2$ whenever $r \neq 4$ and $s \neq 3$. Let $r = 4$ and $s = 3$. Then $hn = 8$, thus m is even, which contradicts that $m = s = 3$. Now, suppose that $|S_-|$ occurs in the 2nd or the 3rd row of Table 1. In this case $m = r - s$, and if $s \neq 1$, then $2|S_-|/p^m \equiv \pm 2 \pmod{p^2}$, contradicting (9). Let $s = 1$. Then $h\frac{n}{2} - 1 = r - 1 = m = h\lceil \frac{n-2}{2} \rceil$. We obtain that $h = 2$ and n is odd. Then $q = p^2 \equiv 1 \pmod{4}$. If $|S_-|$ is equal to a number in the 2nd row, then by (4), $p^{n+1} - p^{n-1} - p^2 + 1 = 2p^{n+1} - 2p^n - 2p + 2$, and if it is equal to a number in the 3rd row, then $p^{n+1} - p^{n-1} - p^2 + 1 = 2p^{n+1} - 2p^n + 2p - 2$. It is easy to see that none of these equations holds for $n \geq 3$ and an odd prime p .

Suppose that the group G is in class (B). We obtain from (7) that $(q, n) = (9, 3)$ or $(9, 4)$. The possibility that $q^n = 9^3$ gives rise to a strongly regular graph on 9^3 points with valency $|S_-| = 288$. These numbers can be excluded by Brouwer’s database

of strongly regular graphs (see <https://www.win.tue.nl/~aeb/graphs/srg/srgtab.html>). As an alternative way to exclude this case, one may use the classification of solvable primitive permutation groups of rank 3 due to Foulser [2]. Let $q^n = 9^4$. Then by [10, Table 13], the corresponding subdegrees are 1440 and 5120. However, these do not match the numbers given in (6).

Finally, suppose that G is in class (C). Then we obtain from (8) that $(q, n) \in \{(9, 3), (25, 3), (81, 3), (27, 4), (9, 6)\}$. By [10, Table 14], the corresponding nontrivial subdegrees are:

q^n	nontrivial subdegrees
3^6	224, 504
5^6	7560, 8064
3^{12}	65520, 465920

However, none of these matches the numbers in (6). The lemma is proved. \square

Remark 1. We would like to note that in our earlier approach we gave a proof of Theorem 2, which also relies on Lemmas 1-3, but instead of invoking Lester's result (Theorem 3), we used the results of Iosevich et al. [4] on maximum point sets with any two of its points being at distance 0. Here we give an outline. Let $\gamma \in \text{Aut}(\mathbb{F}_q^n)$ be an integral automorphism which fixes the zero vector. We need to prove that γ is a semilinear transformation. By the fundamental theorem of projective geometry we are done if we show that γ preserves both the point and the line set of the projective space $\text{PG}(n-1, q)$. Let us consider the nonsingular quadric \mathcal{Q} of $\text{PG}(n-1, q)$ induced by the quadratic form $x_1^2 + \cdots + x_n^2$. A projective subspace of maximum dimension on \mathcal{Q} is called a *generator* (cf. [5, Chapter 22]). Observe that any subspace U of \mathbb{F}_q^n corresponding to a generator has the property that any two of its points are at distance 0. It follows from [4, Theorem 2 and Lemma 4] that U is a maximum point set with the latter property, and thus γ maps U to a subspace. The latter subspace is contained in S_0 , see Lemma 3, and we conclude that γ permutes the generators among themselves. This observation and the fact that any point of \mathcal{Q} can be expressed as the intersection of some generators yield that γ preserves the set of points on \mathcal{Q} . Then, using Lemma 2, we find that any line of $\text{PG}(n-1, q)$ through two points of \mathcal{Q} is mapped by γ to a line. If $(n, q) \neq (3, 3)$, then any point of $\text{PG}(n-1, q)$ can be expressed as the intersection of two lines each connecting two points of \mathcal{Q} , and this with the previous observation yield that γ preserves the point set of $\text{PG}(n-1, q)$. Then, using again Lemma 2, we conclude that γ preserves the line set of $\text{PG}(n-1, q)$ as well.

Finally, we would like to mention that Theorem 3 was rediscovered by Vroegindewey [13] (the special case when the associated quadratic form has Witt index 1 was treated already in [12]).

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